

Semitopological Q-Rings*

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(November 28, 2001)

Semitopological Vortices (Q-Rings) are identified to be classical soliton configurations whose stability is attributed to both topological and nontopological charges. We discuss some recent work on the simplest possible realization of such a configuration in a scalar field theory with an unbroken $U(1)$ global symmetry. We show that Q-Rings correspond to local minima of the energy, exhibit numerical solutions of their field configurations and derive virial theorems demonstrating their stability.

As we celebrate the 60th birthday of Holger Bech Nielsen we can without doubt assess his contributions to the development of the theory of strings and vortices to bear the strongest possible impact. Indeed his early work on the development of multiparticle dual models [1–3] was soon after followed by the introduction of the string picture in the study of strong interaction physics [4]. At the time it improved tremendously our physical understanding of dual models [5]. The string concept, of course, was bound to become much more useful in the unification of particle interactions with gravity. Aside from Holger’s contribution to the development of the “string idea” he much later provided the first covariant formulation of a vortex in a theory with spontaneously broken abelian gauge symmetry [6]. The stability of such a gauged vortex is due the presence of a topological charge. The cosmic role of such topological defects in the phase transitions of the early universe has been important. It is in the spirit of this line work of Holger’s that we will present a novel class of vortex like configurations that share some of the properties of topological solitons as well as those that are non-topological in character. Hence their identification as semitopological. The work was done in collaboration with E.G.Floratos, S.Komineas and L.Perivolaropoulos [7]

Non-topological solitons (Q balls) are localized time dependent field configurations with a rotating internal phase and their stability is due to the conservation of a Noether charge Q [9]. They have been studied extensively in the literature in one, two and three dimensions [8]. In three dimensions, the only localized, stable configurations of this type have been assumed to be of spherical symmetry hence the name Q balls. The generalization of two dimensional (planar) Q balls to three dimensional Q strings leads to loops which are unstable due to tension.

Closed strings of this type are naturally produced during the collisions of spherical Q balls and have been seen to be unstable towards collapse due to their tension [10,11].

There is a simple mechanism however that can stabilize these closed loops. It is based on the introduction of an additional phase on the scalar field that twists by $2\pi N$ as the length of the loop is scanned. This phase introduces additional pressure terms in the energy that can balance the tension and lead to a stabilized configuration, the *Q ring*. This type of pressure is analogous to the pressure of the superconducting string loops [12] (also called ‘springs’ [13]). In fact it will be shown that Q rings carry both Noether charge and Noether current and in that sense they are also superconducting. However they also differ in many ways from superconducting strings. Q rings do not carry two topological invariants like superconducting strings but only one: the winding N of the phase along the Q ring. Their metastability is due not only to the topological twist conservation but also due to the conservation of the Noether charge as in the case of ordinary Q balls. Due to this combination of topological with non-topological invariants Q rings may be viewed as semitopological defects. In what follows we demonstrate the existence and metastability of Q rings in the context of a simple model. We use the term ‘metastability’ instead of ‘stability’ because *finite size* fluctuations can lead to violation of cylindrical symmetry and decay of a Q ring to a Q ball as demonstrated by our numerical simulations.

Consider a complex scalar field whose dynamics is determined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^* \partial^\mu \Phi - U(|\Phi|) \quad (1)$$

The model has a global $U(1)$ symmetry and the associ-

*Contributed to the 4th Workshop “What Comes Beyond The Standard Model”, Bled, Slovenia, July 17-27 2001 in honor of Holger Beck Nielsen’s 60th Birthday.

ated conserved Noether current is

$$J_\mu = Im(\Phi^* \partial_\mu \Phi) \quad (2)$$

with conserved Noether charge $Q = \int d^3x J_0$. Provided that the potential of (1) satisfies certain conditions [9,8] the model accepts stable Q ball solutions which are described by the ansatz $\Phi = f(r)e^{i\omega t}$. The energy density of this Q ball configuration is localized and spherically symmetric. The stability is due to the conserved charge Q .

In addition to the Q ball there are other similar stable configurations with cylindrical or planar symmetry but infinite, not localized energy in three dimensions. For example an infinite stable Q string that extends along the z axis is described by the ansatz

$$\Phi = f(\rho)e^{i\omega t} \quad (3)$$

where ρ is the azimuthal radius. This configuration has also been called ‘planar’ or ‘two dimensional’ Q ball [8].

The energy of this configuration can be made finite and localized in three dimensions by considering closed Q strings. These configurations which have been shown to be produced during spherical Q ball collisions [10,11] are unstable towards collapse due to their tension. In order to stabilize them we need a pressure term that will balance the effects of tension. This term appears if we substitute the string ansatz (3) by the ansatz of the form

$$\Phi = f(\rho)e^{i\omega t}e^{i\alpha(z)} \quad (4)$$

where $\alpha(z)$ is a phase that varies uniformly along the z axis. This phase introduces a new non-zero J_z component to the conserved current density (2). The corresponding current is of the form

$$I_z = \int dz \frac{d\alpha}{dz} 2\pi \int d\rho \rho f^2 \quad (5)$$

Consider now closing the infinite Q string ansatz (4) to a finite (but large) loop of size L . The energy of this configuration may be approximated by

$$\begin{aligned} E &= \frac{Q^2}{4\pi L \int d\rho \rho f^2} + \pi L \int d\rho \rho f'^2 \\ &+ \frac{(2\pi N)^2 \pi}{L} \int d\rho \rho f^2 + 2\pi L \int d\rho \rho U(f) \\ &\equiv I_1 + I_2 + I_3 + I_4 \end{aligned}$$

where we have assumed $\alpha(z) = \frac{2\pi N}{L}z$ and the terms I_i are all positive. Also Q is the charge conserved in 3D defined as

$$Q = \omega 2\pi L \int d\rho \rho f^2 \quad (6)$$

The winding $2\pi N = \int dz \frac{d\alpha}{dz}$ is topologically conserved and therefore the current (5) is very similar to the current of superconducting strings.

After a rescaling $\rho \rightarrow \sqrt{\lambda_1}\rho$, $z \rightarrow \lambda_2 z$ the rescaled energy may be written as

$$E = \frac{1}{\lambda_1 \lambda_2} I_1 + \lambda_2 I_2 + \frac{\lambda_1}{\lambda_2} I_3 + \lambda_1 \lambda_2 I_4 \quad (7)$$

This configuration can be metastable towards collapse since Derrick’s theorem [14] is evaded due to the time dependence [15,16] of the configuration (4). Demanding metastability towards collapse in any direction we obtain the virial conditions

$$I_3 + I_4 = I_1 \quad (8)$$

$$I_2 + I_4 = I_1 + I_3 \quad (9)$$

In order to check the validity of these conditions numerically we must first solve the ode which f obeys. This is of the form

$$f'' + \frac{1}{\rho} f' + (\omega^2 - (2\pi N)^2/L^2)f - U'(f) = 0 \quad (10)$$

with boundary conditions $f(\infty) = 0$ and $\frac{df}{d\rho}(0) = 0$. Equation (10) is identical with the corresponding equation for 2D Qballs [16] (see ansatz (3)) with the replacement of ω^2 by

$$\omega^2 - \frac{(2\pi N)^2}{L^2} \equiv \omega'^2 \quad (11)$$

Solutions of (10) for various ω' and $U(f) = \frac{1}{2}f^2 - \frac{1}{3}f^3 + \frac{B}{4}f^4$ with $B = 4/9$ were obtained in Ref. [16]. Now it is easy to see that the first virial condition (8) may be written as

$$\omega'^2 \int d\rho \rho f^2 = 2 \int d\rho \rho U(f) \quad (12)$$

This is exactly the virial theorem for 2D Qballs (infinite Q strings) with $N = 0$ and field ansatz given by (3) with ω replaced by ω' . The validity of this virial condition has been verified in Ref. [16]. This therefore is an effective verification of our first virial condition (8).

The second virial condition (9) can be written (using the first virial (8)) as

$$2I_3 = I_2 \quad (13)$$

which implies that

$$\frac{2\pi N^2}{L^2} = \frac{\int d\rho \rho f'^2}{\int d\rho \rho f^2} \quad (14)$$

This can be viewed as a relation determining the value of L required for balancing the tension ie for metastability.

These virial conditions can be used to lead to a determination of the energy as

$$E = 2(I_1 + I_3) \quad (15)$$

In the thin wall limit where $2\pi \int d\rho \rho f^2 = Af_0^2$ (A is the surface of a cross section of the Q ring) this may be written as

$$E \simeq \frac{Q^2}{2LAf_0^2} + \frac{(2\pi N)^2 Af_0^2}{2L} \quad (16)$$

and can be minimized with respect to f_0^2 . The value of f_0 that minimizes the energy in the thin wall approximation is

$$f_0 = \sqrt{\frac{Q}{2\pi NA}} \quad (17)$$

Substituting this value back on the expression (16) for the energy we obtain

$$E = \frac{2\pi NQ}{L} \quad (18)$$

This is consistent with the corresponding relation for spherical Q balls which in the thin wall approximation lead to a linear increase of the energy with Q .

The above virial conditions demonstrate the persistence of the Q ring configuration towards shrinking or expansion in the two periodic directions of the Q ring torus for large radius. In order to study the Q rings of any size and its stability properties towards any type of fluctuation we must study the full evolution of a Q ring in 3D by performing energy minimization and numerical simulation of evolution. This is precisely what we did for a potential energy given by :

$$U(\phi) = \frac{1}{2}|\Phi|^2 - \frac{1}{3}|\Phi|^3 + \frac{B}{4}|\Phi|^4 \quad (19)$$

The ansatz we used that captures the above mentioned properties of the Q ring is

$$\Phi = f(\rho, z) e^{i[\omega t + n\phi]} \quad (20)$$

where the center of the coordinate system now is in the center of the torus that describes the Q ring and the ansatz is valid for *any* radius of the Q ring. We have also replaced N by n .

The energy of this configuration is

$$E = \frac{1}{2} \frac{Q^2}{\int f^2 dV} + \frac{1}{2} \int \left[\left(\frac{\partial f}{\partial \rho} \right)^2 + \frac{n^2 f^2}{\rho^2} \right] dV + \frac{1}{2} \int \left[\left(\frac{\partial f}{\partial z} \right)^2 \right] dV + \int U(f) dV \quad (21)$$

The field equation for Φ is

$$\ddot{\Phi} - \Delta \Phi + \Phi - |\Phi|^2 \Phi = 0 \quad (22)$$

Substituting the ansatz (20) we find that $f(\rho, z)$ should satisfy

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} - \frac{n^2 f}{\rho^2} + \frac{\partial^2 f}{\partial z^2} + (\omega^2 - 1)f + f^2 - Bf^3 = 0 \quad (23)$$

In order to solve this equation we minimized the energy (21) at fixed Q using the algorithm

$$\frac{\partial f}{\partial \tau} = -\frac{\delta E}{\delta f} \Rightarrow \quad (24)$$

$$\begin{aligned} \frac{\partial f}{\partial \tau} = & \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} - \frac{n^2 f}{\rho^2} + \frac{\partial^2 f}{\partial z^2} \\ & + (\omega^2 - 1)f + f^2 - Bf^3 \end{aligned} \quad (25)$$

with boundary conditions $f(0, z) = 0$, $\frac{\partial f(\rho, z)}{\partial z}|_{z=0} = 0$. The validity of the algorithm is checked by

$$dG/d\tau = \delta E/\delta f \, df/d\tau = -(dE/d\tau)^2 < 0 \quad (26)$$

In (24) ω is defined as

$$\omega = \frac{Q}{\int f^2 dV} \quad (27)$$

In the algorithm, we have used the initial ansatz:

$$f(\rho, z) = \text{const} \exp^{-\frac{(\rho - \rho_0)^2 + z^2}{\text{const}}} \quad (28)$$

where ρ_0 is a fixed initial radius. The energy minimization resulted to a non-trivial configuration $f(\rho, z)$ for a given set of parameters B, n, Q in the expression for the energy. We then used (27) to calculate ω and constructed the full Q ring configuration using (20). While the details of our numerical analysis can be found elsewhere we just report the main results.

It was verified that the Q ring configurations evolve with practically no distortion and are metastable despite their long evolution. Finite size nonsymmetric fluctuations were found to lead to a break up and eventual

decay of the Q ring to one or more Q balls. Thus a Q ring is a metastable as opposed to a stable configuration.

The Q ring configuration we have discovered is the simplest metastable ring-like defect known so far. Previous attempts to construct metastable ring-like configurations were based on pure topological arguments (Hopf maps) and required gauge fields to evade Derrick's theorem due to their static nature [17,18]. This resulted in complicated models that were difficult to study analytically or even numerically. Q rings require only a single complex scalar field and they appear in all theories that admit stable Q balls including the minimal supersymmetric standard model (MSSM). The simplicity of the theory despite the non-trivial geometry of the field configuration is due to the combination of topological with non-topological charges that combine to secure metastability without added field complications.

The derivation of metastability of this configuration opens up several interesting issues that deserve detailed investigation. They pertain to the various mechanisms of formation of Q Rings (Kibble and Affleck-Dine mechanisms, Q ball collisions etc.) as well as on the dependence of the winding N on Q. We hope to have something interesting to report in the forthcoming 70th birthday celebration of Holger.

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